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Mathematics Teachers' Understanding of the Algorithms for Solving Quadratic Function Problems

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Abstract

This study investigated secondary school mathematics teachers' understanding of the mathematical concepts underlying the algorithms used in teaching quadratic functions. It examined three constructs: i) why the graph of a parabola is a curve and not a straight line; ii) why the solution of a quadratic equation remains unchanged when the equation is multiplied by -1; and iii) the justification for the counter-intuitive shift of the parabola in horizontal transformations. A qualitative case study design was employed, involving five secondary school mathematics teachers selected through a combination of purposive and convenience sampling. Data were collected using a Subject Matter Knowledge Questionnaire (SMKQ) that focused on teachers' understanding of quadratic functions and their underlying algorithms. Responses were analysed through conventional qualitative content analysis. Findings revealed that while most teachers demonstrated adequate procedural skills, they exhibited limited conceptual understanding of the meaning underlying the algorithms in the constructs investigated. Their reasoning often reflected a focus on knowing how rather than understanding why, indicating a gap between procedural fluency and conceptual depth. The study concludes that strengthening teachers' conceptual understanding of mathematical algorithms is essential for improving instructional quality and learner outcomes. It recommends targeted professional development programmes that integrate both procedural and conceptual aspects of mathematical knowledge.

Keywords: quadratic functions, knowledge of meaning behind algorithms, subject matter knowledge

INTRODUCTION

The assertion that subject matter knowledge is an indispensable component of teachers' knowledge base for teaching is widely recognised and supported by numerous studies (Copur-Gencturk & Tolar, 2022; Ning et al., 2024; Nixon & Bennion, 2025;

Nyaaba & Zhai, 2024). This is unsurprising, as a teacher's primary role, regardless of their orientation towards mathematics, its teaching methods, or curriculum initiatives, is to facilitate student learning in a specific subject (Cahyono & Rusiadi, 2025; Gautam & Agarwal, 2023; Siregar et al., 2025). Therefore, for teachers to be effective, they must possess a solid understanding of the content they teach and what students are expected to learn. This understanding influences every aspect of the teaching process, including explanations of concepts, questions posed during teaching, feedback provided to students, and how teachers address students' errors (Lindström et al., 2025; Shulman, 1986).

In South Africa, there has been a concerning decline in the number of students taking mathematics in school certificate examinations over the past decade. Many students opt for Mathematical Literacy, which is perceived to be easier to pass but offers fewer opportunities for those pursuing careers in fields that require mathematics (Department of Basic Education [DBE], 2023; DBE, 2024; Reddy et al., 2016). All Grade 12 students must choose between a more rigorous Mathematics course and the less challenging Mathematical Literacy, which primarily equips them with basic numeracy skills. Furthermore, students continue to perform poorly in mathematics compared to subjects such as physical sciences, accounting, and geography, which have shown improvement (DBE, 2023; DBE, 2024; Volmink, 2020). Some stakeholders attribute this lack of interest and poor performance to the nature of teachers' mathematical work in the classroom (Venkat & Spaull, 2015). One area of particular concern is the learning of quadratic functions. Diagnostic reports from mathematics examinations in the National Senior Certificate indicate that students struggle with questions related to quadratic functions and have difficulties in demonstrating conceptual understanding of mathematical algorithms (DBE, 2022; DBE, 2023; DBE, 2024). This is alarming, given that functions are fundamental to students' mathematics education (Schliemann et al., 2021; Thompson & Carlson, 2017). Mastery of quadratic functions exposes students to various mathematical concepts, including definitions, multiple representations, solution strategies, precision, and the proper use of language. Suominen (2018) emphasised that knowledge of quadratic functions is essential for studying calculus, modelling, and algebraic concepts. Despite the importance of understanding quadratic functions within the school mathematics curriculum, students face numerous challenges. These include difficulties in graphing parabolas and misunderstanding the relationship between quadratic functions and quadratic equations (Tekim et al., 2009; Beki et al., 2010, both cited in Memnun et al., 2015). Moreover, research into teachers' mathematical knowledge specifically related to teaching quadratic functions is limited (Aziz et al., 2018; Febriani et al., 2024; Newton et al., 2022; Ubah & Bansilal, 2018; Yaşa, 2022). Among these studies, only a few, such as that by Yaşa (2022), have explored the subject matter knowledge of practising teachers regarding quadratic equations, functions, and graphs.

Literature review

According to Shulman (1986), subject matter content knowledge encompasses both the amount and organisation of knowledge that a teacher possesses. He argued that SMK should include knowledge of facts and concepts within a subject area as well as an understanding of its underlying structures. The structures of a subject consist of substantive and syntactic dimensions. The substantive structure refers to knowledge of facts, concepts, principles, and their interrelationships, while the syntactic structure pertains to the methods used to establish what is valid within the discipline. Shulman further conceptualised SMK as comprising both

knowledge that something is so and knowledge of why it is so. "Knowing that" refers to teachers' understanding of procedures, algorithms, rules, and concepts within specific curriculum topics, whereas "knowing why" involves understanding the meanings and justifications underlying those concepts and procedures. Teachers, therefore, should not only be able to identify what is acceptable within a concept but also justify why a statement is true and explain how it connects to other concepts within the subject area and beyond.

In the context of mathematics education, Even and Tirosh (1995) argued that "*knowing that* involves declarative knowledge of rules, algorithms, procedures, and concepts related to specific mathematical topics in the school curriculum" (Even & Tirosh, 1995, p. 9). They emphasised that this form of knowledge is essential because it forms the foundation for sound pedagogical content knowledge (PCK), influencing the kinds of questions and activities teachers use in teaching. Conversely, *knowing why* includes "knowledge which pertains to the underlying meaning and understanding of why things are the way they are, enables better pedagogical decisions" (Even & Tirosh, 1995, p. 9). It also influences teachers' decision-making in the classroom and how they present the topic to students. Even and Tirosh (1995) further pointed out that, although "knowing that" is an essential aspect of teachers' SMK, making good pedagogical decisions requires teachers to have the *"knowing why"* aspect of their topic.

Hiebert and Lefevre (1986) similarly characterised mathematical knowledge as consisting of procedural and conceptual components. Procedural knowledge refers to the understanding of how something is done, encompassing definitions, rules, theorems, and algorithms used to solve mathematical problems. Scholars such as Canobi (2009) and Rittle-Johnson and Schneider (2014) describe procedural knowledge as the ability to follow specific steps sequentially to solve mathematical tasks. According to the New York State Education Department (2005), procedural fluency involves executing operations flexibly, efficiently, and appropriately. In contrast, conceptual knowledge concerns understanding why something is true. It is distinguished by its richness in relationships and can be viewed as integrated knowledge that connects rules, facts, principles, and generalisations into a coherent whole (Groth & Bergner, 2006).

Similarly, Ball (1991) distinguished between knowledge of mathematics and knowledge about mathematics as two critical components of mathematics teachers' SMK. Knowledge of mathematics, also referred to as substantive knowledge, encompasses both procedural and conceptual understanding and concerns the meaning, connectedness, and correctness of mathematical ideas. Teaching for meaningful understanding, therefore, requires teachers to possess a deep grasp of the underlying meanings of the mathematical concepts and procedures they teach. In contrast, knowledge about mathematics involves understanding how truth or validity is established within the discipline.

From the literature reviewed above, it is evident that teachers' knowledge of the meaning behind algorithms constitutes an essential aspect of mathematics teachers' SMK. However, research examining teachers' understanding of the meanings underlying algorithms remains limited, particularly in the context of quadratic functions. This study aims to address this gap and contribute to a deeper understanding of how secondary school mathematics teachers interpret the meanings behind the algorithms they use when teaching quadratic functions. The concept of quadratic functions is widely taught in school mathematics across different countries. The guiding research question is therefore: How do secondary school mathematics teachers understand and interpret the meanings behind the algorithms they use when teaching quadratic functions?

METHODS

This section presents the research methodology, including the research design, participants, data collection instruments, procedures, and data analysis techniques employed in the study. The study employed the qualitative descriptive case study design (Tobin, 2010). We adopted a qualitative approach in this study because our intention was to conduct an in-depth inquiry into the construct of interest: teachers' knowledge of the meaning behind algorithms. This approach provided us with the opportunity to describe the participants' subject matter knowledge, which cannot be adequately addressed using a quantitative approach. For instance, a numerical analysis of a content test cannot reveal teachers' reasoning in a question. The case study enables the researcher to explore and describe each participant in relation to the constructs investigated in a given study. It allows each participant to be analysed and compared with the other participants. From the comparison, similarities (and variants) are explored. According to Stake's (1994) categorization, this study is an intrinsic case study, as it sought to gain a better understanding of the teachers' SMK and was not used for theory building.

Participants

The study was conducted with a sample of five practicing teachers who were purposively and conveniently selected from a pool of twenty-one teachers who indicated interest in participating in the study. Probability sampling is not necessary in this study, as it is qualitative. Participants were selected because of the proximity of their schools to the researchers. Data is easily collected if the schools of the participants are not too far from the researchers. In making the selection, we tried to balance gender, qualifications, and years of experience of the participants. The real names of the participants were not used for ethical reasons. The participants read and understood the research objectives, the extent of data to be collected, and their right to withdraw from the study at any point if their welfare was threatened by the study. They signed the consent form, indicating that they had voluntarily accepted participation in the study without being coerced. The participants' brief background is presented as follows.

Thembi

Thembi is a female teacher with three years of experience teaching Grade 11 mathematics. She holds a Bachelor of Science degree in Mathematics and a Postgraduate Certificate in Education (PGCE) in Mathematics and Natural Sciences.

Tony

Tony is a male teacher with 21 years of experience teaching Grade 11, making him one of the most experienced educators in the group. He holds a Diploma in Education Studies and an Advanced Certificate in Education (ACE) in Mathematics.

Aneta

Aneta is a female teacher with eight years of experience teaching Grade 11. She holds a Bachelor of Education (B.Ed.) degree, an integrated qualification that combines subject content knowledge with pedagogical training.

Thabo

Thabo is a male teacher with 16 years of experience teaching Grade 11 mathematics. He holds a Diploma in Education, an Advanced Certificate in Education (ACE), and a Bachelor of Education honours degree in Mathematics.

Ken

Ken is a male teacher with 20 years of experience teaching Grade 11. He holds a Diploma in Education, a Bachelor of Technology (B.Tech.) degree, and an Advanced Certificate in Education (ACE).

Data Collection Instrument

The instrument used for data collection in this study was a Subject Matter Knowledge Questionnaire (SMKQ). The instrument was a content-based self-report questionnaire. The questionnaire required teachers to explain, in writing, some of the algorithms they use in teaching quadratic functions. In each question, a scenario was provided. Teachers were asked to explain the meaning/reason(s) behind the given algorithm or skill in the question item. The scenarios are the type of questions (problems) or situations that teachers can encounter in the classroom. The question items are provided in the appendix. SMKQ was developed by the researchers. The three items in the questionnaire were based on some of the concepts in quadratic functions, which students find problematic (having misunderstandings or conceptual difficulties), as indicated in the literature. For instance, students often struggle with drawing parabolas. Students draw graphs of quadratic functions as if they were linear functions (Tekim et al., 2009, in Memnun et al., 2015). Teachers' understanding of this difficulty was investigated in item 1. Question item 2 was based on students’ difficulties in understanding how the quadratic function relates to a quadratic equation (Beki et al., 2010, in Memnun et al., 2015). Question item 3 was based on students having difficulties in understanding the counterintuitive shift in the horizontal translation of the parabola (Zakis et al., 2003). To ensure the instrument's validity, the questionnaire was sent to three mathematics subject specialists in the Department of Education and two researchers in the field of mathematics education for validation purposes. Each of them confirmed that the items in the questionnaire could dig into teachers' knowledge of the meaning of the concepts investigated. Inputs from the mathematics specialists and researchers were used to refine the questionnaire. We administered the questionnaire to the participants in their schools after we secured permission to enter the site from the Department of Education and the school governing body (SGB). The participants responded to the questionnaire in the presence of the researchers. Each participant responded to the questionnaire at their own pace. The participants took from 1 hour 30 minutes to 2 hours to complete the questionnaire. It took one week to administer the questionnaire to all the teachers.

Trustworthiness

Although the rules for designing qualitative studies differ from those for studies using quantitative methods, the difference does not warrant a compromise in the rules of rigour. In this study, trustworthiness was achieved by following the ideas of Lincoln and Guba (1986). We took the following steps to ensure that our study is of high quality. The procedures used in data collection, analysis, and interpretation in this study were similar to those employed in other successful studies. We employed triangulation to enhance confidence in the conclusions drawn from the data. The form of triangulation employed in this study was utilising a wide range of participants (Shenton, 2004). Specifically, in this research study, five teachers from differing backgrounds were compared to provide a thick description of the subject matter knowledge investigated. We provided an in-depth description of the methodology followed to allow other scholars to scrutinise the integrity of the findings. We took steps to ensure that the findings accurately represent the ideas and experiences of the participants, rather than our own bias and preferences, as we used raw data in the analysis and reporting. Findings from the study cannot be generalised. This is a problem with qualitative studies.

Data Analysis

The data analysis approach employed in this study is the qualitative conventional content analysis (Hsieh et al., 2005). Qualitative content analysis is a systematic and objective method for describing and analysing phenomena. Conventional content analysis is typically employed in studies designed to describe a phenomenon. In this method of data analysis, categories and names emerge from the raw data, rather than being preconceived. The process in content analysis involves three main phases: preparation, organisation, and reporting of results. The preparation phase involves collecting data for content analysis, interpreting the data, and determining the unit of analysis. The organisation phase includes open coding, creating categories, and abstraction. In the reporting phase, results are described in terms of the categories that describe the phenomenon (Elo et al., 2014). In our analysis, we followed all three phases of analysis. At the preparation phase, we repeatedly studied the explanations provided by each teacher to make sense of the data. From this, we identified the units of analysis. In the second phase of our analysis, we compared and categorised teachers' responses by using the level-specific knowledge necessary for answering each question. From the comparison, we were able to identify similarities and differences in the teachers' knowledge. The level-specific knowledge required to answer each question is presented in the following paragraph. This analytic tool was created by the researchers and was validated by experts in the field of mathematics education. In phase three, we reported the content of the teachers' knowledge of the meaning behind the algorithms or procedures in each question.

Level-specific knowledge needed in responding to the items

Question 1

On Question 1.1, the misconception emanates from learners' understanding of drawing graphs of linear functions, where a straight line is drawn using the plotted points. The linear function is the first family of functions the learners are exposed to in the South African school curriculum.

On Question 1.2, the teacher must recognise that there are an infinite number of points between any two points on the parabola. By taking smaller intervals between two points on the x-axis of the parabola (say 0 and 1), the teacher can draw a graph for the two points, which yields a curve.

Question 2

On Question 2.1, the student was correct to reason that $g(x) = - x^2 + 5x + 6$ and $f(x) = x^2 - 5x – 6$ are not the same function because for given values of x, the function values of $g(x) = - x^2 + 5x + 6$ are not the same as that of $f(x) = x^2 - 5x – 6$ except at $y = 0$, (the zeros of the function or the x-intercept if the graph is drawn).

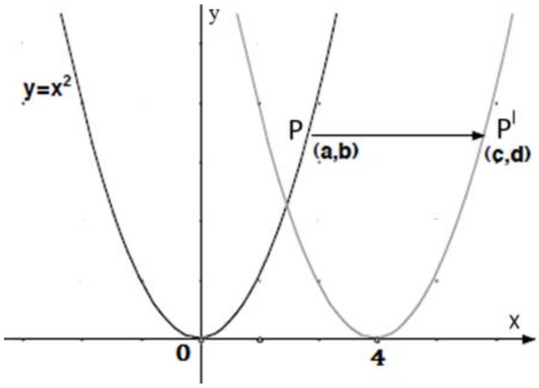
On Question 2.2, equating to zero means that one is looking for the x-intercept, equating each of $- x^2 + 5x + 6$ and $x^2 - 5x – 6$ to zero, and solving the result will yield the same answer (same x-intercept). The use of the graphs of the two functions (geometric approach) on the same set of axes as will clearly show the similarities and differences between the two functions.

Question 3

First, it should be noted that students' difficulty in understanding the horizontal shifting of the parabola is legitimate because the graph moves in an unexpected direction, which seems counterintuitive to the rule of translation learned in lower grades. The problem is traceable to the traditional approach in the school mathematics curriculum, whereby the horizontal transformation of the parabola is presented in the context of a function rather than as a translation (Zazkis, 2010; Zazkis et al., 2003).

Understanding the translation of a parabola in the context of translation rather than function (Zazkis, 2010; Zazkis et al., 2003) will aid in resolving the seemingly counterintuitive movement of the graph after horizontal translation of the parabola. For clarity, this approach is explained below:

Below is the graphical illustration for the translation of the parabola $y = x^2$, 4 units to the right, i.e., $P(x; y) \longrightarrow P^I(x + 4 ; y)$.



Our aim was to find the algebraic representation of the translation shown above.

We have to note that the points in the object are given by $y = x^2$.

By focusing on P(a, b) and its image P^I(c, d),

$d = b$, and $c = a + 4$, $\Rightarrow a = c - 4$.

To find an equation connecting c to d

$b = a^2$ [(a, b) is a point on the object described by $y = x^2$]

$\therefore d = (c - 4)^2$. (This is true for every point in the image since every point in the image is shifted 4 units to the right.

Hence, $y = (x - 4)^2$. This explains the counterintuitive appearance of "- 4" in the horizontal, translation to the right.

RESULT AND DISCUSSION

Participants' Responses to Question 1

On Question 1, participants were asked to explain in detail how they could convince their students that any two points on a parabola should be connected by a curve, not a straight line. It should be noted that when plotting a parabola point by point, students often join two points with a straight line, even after the teacher has told them that the points in a parabola form a curve. Students seem not to understand why points on a parabola must be joined with a curve, because from their experience with linear functions, they are used to joining points on a graph with straight lines. Teachers were asked to explain in detail how they could teach students to understand that any two points on the parabola form a curve (see Question 1, SMKQ). The participants' responses and an analysis of their responses are as follows

Question 1

Thembi: I would ask the learner to make a table of values that includes ½, e.g., 0, ½, 1, 1½, 2, 2½, etc., for x. I will ask them to plot the graph. On the same set of axes, I will ask them to plot with ¼, e.g., 0, ¼, ½, 1, 1¼, 1½, 1¾, 2 for x. Then, they will be able to see that the graph gets smoother and smoother as we reduce it from ½, to ¼, to 1/8, etc. I will then explain that there are many points between any two points (as many as uncountable). Therefore, as the points increases, the graph becomes smoother and smoother until it is continuous.

Thembi's explanation indicates that she understands why any two points on a parabola should be joined with a curve and not a straight line. Her explanation that there are an infinite number of points between any two points on the parabola and that plotting those numerous points will yield a smoother curve can convince her students that any two points on a parabola are part of a curve. Following Thembi's explanation, a graph of a portion of the parabola between $x = 0$ and $x = 1$ yields a curve, as shown in Figure 1.

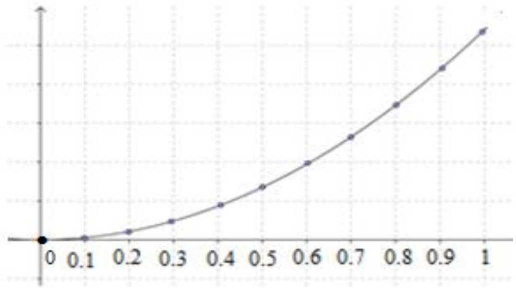


Figure 1. Graph showing the curve nature of a parabola between x = 0 and x = 1

Tony: They must know that a parabola has a line of symmetry and should be drawn using freehand and not a ruler. They should also know that there will be two values of the y-coordinate for any value of the x-coordinate. They must know that it is a graph of the second degree, so it must have one turning point, which is a curve.

The explanation given by Tony on how he can convince students that any two points on a parabola should be joined with a curve and not a straight line strongly indicates that Tony has a challenge in providing a convincing explanation of why any two points on a parabola should be joined with a curve and not a straight line. The "line of symmetry" does not apply to curves only; a line segment has a line of symmetry. Knowing that the parabola "is a graph of the second degree," as presented by Tony, cannot convince a student or any adult mathematician that a parabola is a curve. That a parabola has a turning point does not make it a curve. After all, two inclined line segments with one end of a line segment touching the end of the other line segment have a turning point. (Tony made a mistake when he indicated that "there will be two values for the y coordinate for a single x-coordinate"). This is not true; the reverse is correct. However, the correct presentation of that statement has no value in arguing that the connection between any two points on a parabola is on a curve.

Aneta: I have to use the concept of gradient between two points on a curve. I will draw tangents and calculate the gradient between various points. I will show that the gradients vary; therefore, we cannot connect points with ruler since it is non-linear function.

Aneta's explanation has not shown that the two points formed a curve. It has only proved that the two points are not collinear. The fact that the two points are not collinear does not mean that they form a curve. In the first place, the tangent Aneta said she would draw, or make use of only one of the two points on the curve, since the tangent will touch the parabola at one point only, while the issue here is about two points on the parabola. In addition, calculating the gradient between two points on a curve and comparing it with the gradient of a tangent drawn from either of these points cannot prove that the connection between any two points is on a curve and not a line. Aneta's explanation suggests that she had difficulties explaining why any two points on the parabola should be connected with a curve and not a straight line.

Thabo: Establish the line of symmetry by obtaining the x-coordinate, then try to obtain the y-value in a curve from the graph drawn as in figure 1a [on the question – see the appendix]*, and it will be established that the corresponding value does not fit into the given function. Hence, through calculation, it can be obtained that the y-value lies below the two points at the bottom of the graph.*

The explanation given by Thabo that two points on a parabola are on a curve is not valid. The explanation shows that Thabo has difficulties in providing the reason why any two points on a parabola should be joined with a curve and not a straight line.

Ken: $y = x^2$ is the locus of points that are equidistant from a certain given centre. The result is a conic shape that is shaped or –shaped. Therefore, the points are not linear but result in a curve.

Using the definition of conics to explain why the connection between two points on a parabola is on a curve and not a straight line is not level-appropriate knowledge. In South Africa, grade 11 students are unfamiliar with conics. If Ken tried this method, the students would get more confused because they had not encountered such a definition before. Conics are not currently studied in the South African school mathematics curriculum. The explanation by Ken indicates that he has an understanding of why any two points on a parabola should be joined by a curve, rather than a straight line, but he lacks the level-specific knowledge of how to explain it to the students.

Participants' Responses to Question 2

Thembi: She was right in saying that the two functions are different from each other. One is concave up, the other is concave down: ∪ *and* ∩.

The solution of the equation of a quadratic function is the same as the x-intercept of the parabola. So for both functions, the intersection with the x-axis coincides at $x = -1$ or $x = 6$ as in Figure 2.

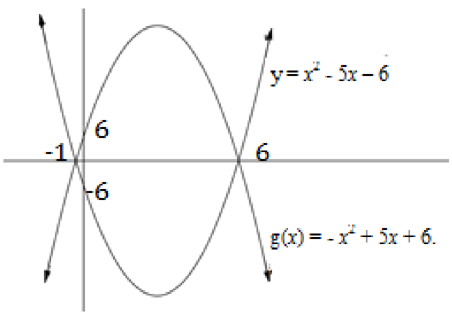


Figure 2. Graph of the two functions showing their intersections

Thembi's explanation shows that she understands that the two functions are not the same. She indicated that the two functions were a reflection of each other along the *x*-axis when she declared that "one is concave up and the other is concave down" and that “for both functions, the intersection with the x-axis coincides”. Thembi’s response indicates that she understands the concept of multiplying a function by -1. By applying a geometrical approach, Thembi proved that although the two functions were not the same, their graphs have the same x-intercepts.

Tony: 1. Yes, she was right.

It can only be right if both g(x) and f(x) are equal to zero because multiplication by zero is always zero, and zero has no sign.

Although Tony knows that the two functions are different, he only demonstrated a limited understanding of the meaning behind multiplying a function by -1. He did not indicate why the functions should be “equal to zero” in the first place. The student rejected the answer from $g(x)$ because of the differences in their coefficients. Tony relied too much on using algebraic means (calculations) to find an answer to the problem, as the only explanation he gave centred on the fact that “multiplication by zero is always equal to zero, and zero has no sign”. Although Tony’s argument was correct, however, he did not explore the multi-representational nature of the quadratic functions in seeking how to convince the student or justify his solution. By using the graphical approach and reflecting on the graph, Tony could have provided an excellent justification for the solution.

Aneta: 1. The student was wrong since g(x) and f(x) are the same. Multiplying all terms with the same variable -1 does change the function.

ii) $-x^2 + 5x + 6 = -(-x^2 - 5x - 6)$

$= [(x-6)(x+1)]$

$0 = -1[(x-6)(x+1)]$

$0 = (x-6)(x+1)$

$x = 6$ or $x = -1$

Therefore, since the solutions of f(x) and g(x) are the same, the two functions are the same.

Aneta’s response suggests that she perceives a function and an equation to be the same thing. This was evident when Aneta declared that “the student was wrong since $g(x)$ and $f(x)$ are the same [$g(x) = -x^2 + 5x + 6$, and she identified $x^2 - 5x - 6$ to be $f(x)$]. Although the solutions of $-x^2 + 5x + 6 = 0$ and $x^2 - 5x - 6 = 0$ are the same, the functions $g(x) = -x^2 + 5x + 6$ and $f(x) = x^2 - 5x - 6$ are not the same. Aneta’s misconception about the difference between the equation of a function and the function itself impeded her ability to find ways of convincing the student that the solution is correct for both $-x^2 + 5x + 6 = 0$ and $x^2 - 5x - 6 = 0$, hence, the functions $g(x) = -x^2 + 5x + 6$ and $f(x) = x^2 - 5x - 6$ have the same x-intercept. Aneta depended only on finding an algebraic solution for the problem. Since functions are inherently multi-representational, Aneta could at least have observed that the functions are not the same as she erroneously indicated. In summary, Aneta faces challenges in providing an explanation to convince the student that the solution of $-x^2 + 5x + 6 = 0$ can be obtained by solving $x^2 - 5x - 6 = 0$.

Thabo: (i) Yes, she’s right.

(ii) The two functions g(x) and f(x) have the same x-intercepts but are not the same.

Thabo was correct that the two functions have the same x-intercepts, but they are not the same. However, his explanation was not detailed enough to convince students that the answer was acceptable.

Ken: (i). $g(x) = -x^2 + 5x + 6$ and $f(x) = x^2 - 5x - 6$ are not the same function although they share the same turning point. f(x) is a reflection of g(x) along the x-axis.
(ii) The solution of $g(x) = 0$ is treated as an equation and not a function, hence multiplying by (-1) would have no effect. While if g(x) is multiplied by (-1), it would change the shape of the graph (i.e, reflection along the x-axis).

Analysis of the response by Ken indicates that Ken has knowledge of the meaning behind multiplying a function by -1 or the equation of a function by -1. When he said that "if a function(x) is multiplied by (- 1), it would change the shape of the graph" [but the x-intercepts will remain the same], Ken demonstrated knowledge that could convince the student to accept the answer.

Participants' Responses to Question 3

Thembi:
I would guide the learner to do point-by-point plotting of the mother graph, $y = x^2$. Then on the same of axes plot $y = (x - 4)^2$. Then, I will ask learners to reach a conclusion on the effect of -4. A few simple and similar problems should iron out the misconception.

Qualitative content analysis of the explanation provided by Thembi suggests that she faced challenges in accounting for the seemingly counterintuitive result obtained through the horizontal shifting of the parabola. Point-by-point plotting was used to carry out the investigation; however, the student questioned the result based on her earlier learning about transformations. Drawing more graphs does not answer the question raised by the student; instead, it will continue to yield results that contradict the rule of transformation. Saying that the student has a misconception because she challenged the result of the horizontal translation of the parabola that moved in an unexpected direction suggests that Thembi does not know that the student was making a genuine claim and that she had no idea how to convince the student that the rule of horizontal transformation is preserved in the horizontal transformation of the parabola. It can be concluded that, although he knows the algorithm (i.e., how to use the point-by-point method to show the horizontal shifting of the parabola), he lacks the understanding of the meaning behind the algorithm involved in the horizontal shifting of the parabola.

Tony: They must know that the graph should move 4 units to the right along the x-axis, and the turning point should be at the point (4, 0).

In his response, Tony only explained the meaning of $f(x - 4)$. He did not account for the counterintuitive result obtained from the translation, which the student challenged. Tony could not explain why the parabola moved in an unexpected direction because he lacked the necessary knowledge for such an explanation.

Aneta: Most learners will shift the graph to the left instead of right. Since most believe that the negative sign means the graph's turning point must be negative.
(ii) f(x – a) means you shift the graph.

Aneta did not provide any meaning behind the algorithm in $f(x - a)$. She only explained the algorithm. Aneta's response indicates that she has difficulties in providing the reason for the counterintuitive movement of the parabola in horizontal translation.

Thabo:
Consider the x-intercept (y =0)
$0 = (x - 4)^2$
$0 = x - 4$
$\therefore x = 4$
Hence, the shift is towards the right. If p is negative in $y = a(x + p)^2 + q$

Thabo's response indicated that he has an idea of the meaning behind the horizontal shifting of the parabola, although he did not follow his idea to a logical conclusion.

Ken:
Minus means movement in the opposite direction, i.e., to the right, while a positive sign means movement to the left of the graph.

The explanation given by Ken suggests that Ken has challenges accounting for the parabola moving in a direction that is not according to the rules of translation.

Summary of the findings

Qualitative content analysis of SMKQ indicated that most teachers had difficulties in providing meaning for the algorithms they use to solve problems involving quadratic functions. On Question 1, for instance, teachers were asked to provide an explanation to convince students that the connection between any two points on a parabola is a curve and not a straight line. Only one of the participants provided a grade-relevant explanation that convinces students why the connection between any two points on a parabola is a curve, not a straight line. A participant used knowledge of advanced mathematics, which secondary school mathematics students in the South African school system may not be familiar with, to explain it. Hence, his explanation cannot convince students why the connection between any two points on a parabola is a curve, rather than a straight line. His explanation may not be relevant to typical grade students in South Africa. Explanations provided by three teachers suggested that they lacked an understanding of why two points on a parabola should be connected by a curve, rather than a straight line.

On Question 2, the qualitative content analysis of the teachers' responses in this item indicated that three of the teachers had difficulties in giving meaning to multiplying a given equation of a parabola by -1. The teachers also demonstrated an over-reliance on the algebraic approach in explaining the meaning of multiplying a quadratic equation by -1. Combining both algebraic and geometric approaches (graphs) in the explanation may allow the students to visualise that the two functions, though different, have the same x-intercepts and the same line of symmetry (See Figure 1).

On Question 3, none of the explanations provided by teachers could account for the parabola moving in an unexpected direction during horizontal translation. The teachers demonstrated knowledge of the horizontal transformation of the parabola. Still, they could not explain that the result obtained from the horizontal transformation agrees with the general rule of transformation.

Discussion

Shulman (1986) identified the teachers' knowledge of meaning behind algorithms (conceptual understanding) as one of the dimensions of teachers' subject matter knowledge. This study investigated how secondary school mathematics teachers understand and interpret the meanings behind the algorithms they use when teaching quadratic functions. The findings reveal that many of the teachers in the study encountered significant difficulties in articulating the meanings underlying the algorithms or skills involved in the constructs presented in each question item. Specifically, teachers struggled to explain why the graph of a parabola is a curve and not a straight line; had difficulties in justifying why the solution of the equation of a quadratic function is the same as when the equation is multiplied by -1; and teachers' difficulties in providing explanations for the counterintuitive shift in the horizontal transformation of the parabola. These difficulties suggest that, while teachers may be able to perform procedural steps, their conceptual grasp of the underlying mathematics is often limited. These findings have implications for the teaching and learning of quadratic functions and mathematics in general in South Africa.

These findings resonate with previous research on teachers' mathematical knowledge. For instance, Malambo et al. (2019) examined the knowledge of pre-service mathematics teachers in the concept of functions through an assessment that required participants to perform calculations, justify their answers, and explain the underlying concepts. Their study revealed that the majority of trainee teachers struggled to justify their responses or provide meaningful explanations for the procedures they applied. Similarly, Agbozo and Fletcher (2020) investigated pre-service teachers' understanding of fraction concepts in Ghana and found that although participants often exhibited strong procedural competence, they lacked conceptual understanding; for example, they could perform operations correctly but were unable to explain why dividing a function by another function produces a particular result. These studies, alongside the findings of the present research, highlight a recurring pattern: teachers may possess procedural knowledge but often lack the deeper conceptual understanding that enables meaningful instruction.

The implications of these findings extend beyond the teachers themselves and have a direct impact on student learning outcomes. Teachers' knowledge forms the foundation of their pedagogical decisions, including how they present concepts, scaffold understanding, and respond to students' questions. When teachers are unable to provide clear explanations for why two points on a parabola are connected by a curve rather than a straight line, students may develop misconceptions or fail to grasp the essential characteristics of quadratic functions (Ibeawuchi & Ngoepe, 2014). Conceptual difficulties at the teacher level thus translate into barriers for learners, limiting their ability to interpret graphs, understand function behaviour, and apply mathematical reasoning in problem-solving contexts.

Furthermore, the difficulties observed in this study highlight the critical distinction between procedural competence and conceptual understanding in mathematics teaching. While procedural knowledge allows teachers to perform calculations or follow algorithmic steps, conceptual knowledge equips them to explain, justify, and connect mathematical ideas, thereby supporting students' deep understanding. The lack of conceptual clarity observed in teachers' responses suggests that their pedagogical content knowledge (PCK) may be constrained, particularly in areas that require interpretation and reasoning about functions, limits, and continuity.

Overall, these findings underscore the importance of strengthening teachers' understanding of the meanings behind the algorithms they teach. Professional development programs, teacher education curricula, and instructional support systems should prioritize opportunities for teachers to engage deeply with both the procedural and conceptual dimensions of mathematics. By fostering a stronger conceptual foundation, teachers will be better equipped to support learners' comprehension, reduce misconceptions, and enhance engagement with complex mathematical topics such as quadratic functions.

CONCLUSION

This study revealed a significant gap between secondary school mathematics teachers' procedural fluency and conceptual understanding of the algorithms underlying quadratic functions. While teachers demonstrated adequate skills in executing procedures, their limited ability to explain why these algorithms work suggests that they may be transmitting rote procedures rather than conveying a meaningful mathematical understanding to learners. This procedural-conceptual divide has important implications for instructional quality, as teachers who focus primarily on "how" rather than "why" may struggle to address learners' questions, correct misconceptions, and foster deep mathematical thinking. The findings underscore the critical need for targeted professional development programmes that deliberately integrate both procedural and conceptual aspects of mathematical knowledge, enabling teachers to bridge this gap and ultimately improve learner outcomes in mathematics education.

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Appendix

Question items used for data collection

Question item 1

You always draw the graph of quadratic functions as curves. Most often students

Join two points on a parabola with a straight line.

1.1 Identify the misconception that led students to connect two points on a parabola with a straight line.

1.2 Explain in detail how you can convince your students that the connection between any two points
on the parabola is a curve, as in Figure 1, NOT straight lines as in Figure 1b.

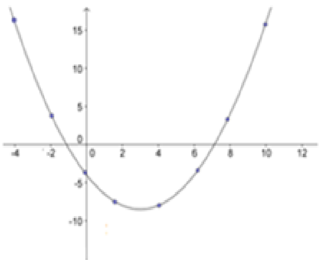

Figure 1a

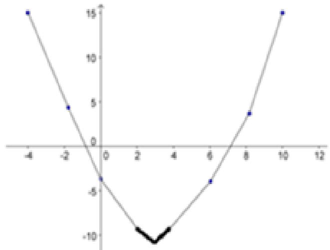

Figure 1b

Question item 2

Your student requested that you help him/her answer the following question:

Given that $g(x) = -x^2 + 5x + 6$. Determine the value of x for which $g(x) = 0$.

You gave your solution in this way:

$-x^2 + 5x + 6 = 0$

Then $x^2 - 5x - 6 = 0$ (multiplying through by -1)

$(x + 1)(x - 6) = 0$

$\therefore$ $x = -1$ or $x = 6$

The student demanded to know why you multiplied through by -1.

She further argued that the answer cannot be accepted since, according to

her $g(x) = -x^2 + 5x + 6$ and $f(x) = x^2 - 5x - 6$ cannot be representing the same function.

2.1 Do you think that the student was right in her argument that $g(x) = -x^2 + 5x + 6$ and $f(x) = x^2 - 5x - 6$ cannot represent the same function?

2.2 How would you justify your solution, since, according to the student, the answer that you obtained is for a different function?

Question item 3

In one of your lessons on "horizontal shifting of the parabola" you drew the graph of $f(x) = x^2$ (Figure 2a). You then used point by point plotting to investigate the result of adding or subtracting a number from x in the function $f(x) = x^2$. You did the investigation by drawing the graph of $g(x) = f(x-4)$, i.e. $g(x) = (x-4)^2$ and you obtained Figure 2b as shown.

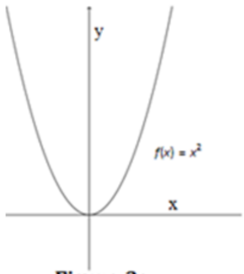

Figure 2a

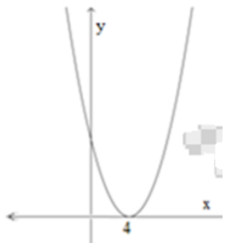

Figure 2b

One of your students questioned the outcome of this investigation. She reasoned that $x - 4$ should lead to a shift to the left and so the graph of g should be the graph of f that has been shifted 4 units to the left in line with the rule of translation of a point. Explain in detail how you can convince students that the rule of horizontal translation of points that students learnt in lower grades is applicable to the horizontal translation of the parabola. Note that the student has rejected the result of the point-by-point plotting.

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